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Breakdown of Hot-Spot model in determining convective amplification in large homogeneous systems

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Convective amplification in large homogeneous systems is studied, both analytically and numerically, in the case of a linear diffraction-free stochastic amplifier. Overall amplification does not result from successive amplifications in small scale high intensity hot-spots, but from a single amplification in a *delocalized* mode of the driver field spreading over the whole interaction length. For this model, the hot-spot approach is found to systematically underestimate the gain factor by more than 50%.

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Much experimental and theoretical work has been devoted over the last two decades to studying the influence of laser beam smoothing on scattering instabilities. Over this period of time, the steady increase of the available laser intensity has rapidly driven the classical perturbative approach to this problem [1] to the wall. In order to meet the new experimental requirements, a simple model has been worked out [2][3] in which the macroscopic reflectivity near and above the instability threshold is assumed to be mainly determined by the high overintensities, or hot-spots (HS), of the laser field, randomly distributed in the interaction region. Surprisingly enough, HS models have only been loosely justified since the work of Ref. [2], mainly on the basis of a somewhat intuitive use of the theory of local maxima of Gaussian fields [4].

The first attempt to a closer look at the validity of the HS models was carried out by one of us in Ref. [5]. Considering the small system limit in which multiple amplifications in successive hot-spots can be neglected, it was shown that random fluctuations of the HS field around its deterministic component can have a significant effect on the overall reflectivity. This was in flat contradiction with the HS model assumption that each hot-spot near its maximum can be approximated by a non-stochastic intensity profile depending on the correlation function of the laser field and being the same for each hot-spot [4][6]. The result of Ref. [5] questioned the validity of HS models in their description of the amplification within each hot-spot *without* challenging the fundamental assumption that it is the hot-spots that determines the overall above-threshold reflectivity. While this assumption is natural for small systems, it cannot be justified in the case of large homogeneous systems.

In this Letter, we address the important question of the conceptual validity of HS models for large homogeneous systems in the simplest case of a one-dimensional (1D) linear convective amplifier. We show both analytically and numerically that the macroscopic reflectivity near and above threshold is determined by the occurrence of intense, well defined, coherent structures of the laser field characterized by a single deterministic profile which is a *delocalized* mode spreading over the whole interaction length. The emerging picture is thus extremely different from successive amplifications in small scale hot-spots, as predicted by the HS approach which fails to retrieve the results of the underlying convective amplifier both quantitatively and qualitatively.

We consider the following one-dimensional (1D) stochastic amplifier

$$\partial_x E(x) = g|S(x)|^2 E(x), \tag{1}$$

with $0 \leq x \leq L$. g is a coupling constant playing the role of the average laser intensity. S is a complex homogeneous Gaussian field defined by $\langle S(x)\rangle = \langle S(x)S(x')\rangle = 0$ and $\langle S(x)S(x')^*\rangle = C(x-x')$, and normalized such that $C(0) = 1$. Solving Eq. (1) with $E(0) = 1$ and averaging over the realizations of S one obtains $\langle E(L)^2\rangle = \prod_{n=1}^{+\infty}(1-2g\kappa_n)^{-1}$, where $\kappa_1 > \kappa_2 ... > \kappa_n > ...$ are the eigenvalues of $C(x-x')$ for $0 \leq x, x' \leq L$, (for the sake of simplicity we assume that κ_n is not degenerate). The smallest g at which $\langle E(L)^2\rangle$ diverges defines the critical coupling g_2 for the intensity. Physically, it corresponds to the threshold for the scattered power [2][7]. It is easily seen that $g_2 = (2\kappa_1)^{-1}$ and one has the asymptotic behavior $\langle E(L)^2\rangle \sim (1-g/g_2)^{-1}$ as $g \uparrow g_2$, As an exact result, this asymptotic behavior provides a good benchmark for testing the validity of the HS model approach to the stochastic amplifier (1).

Denote by l_c the correlation length of S defined by the expansion $|C(x)|^2 = 1 - x^2/l_c^2 + O(x^4/l_c^4)$. The HS models associated with Eq. (1) consist of $N = 1 + \mathrm{Int}(L/2l_c)$ successive boxes of length $L/N \sim 2l_c$ with at most one hot-spot in each box. The hot-spot contribution to $\langle E(L)^2\rangle$ of each box can be computed in the frame of the model of Ref. [5], with the only difference that the hot-spot field is now complex [8]. The simplest (and only tractable) version assumes that the hot-spots are statistically independent. In this case, $\langle E(L)^2\rangle_{HS}$ is given by the average over M of A_{HS}^M where M is the number of hot-spots in the system and A_{HS} is the average amplification for one hot-spot [5][8]. By HS statistical independence, M is a binomial random variable with parameters N and $p = N_>(3)/N$, where $N_>(3)$ is the average number of hot-spots with intensity $I > 3\langle I\rangle$. After some straightforward algebra, one obtains the asymptotic behavior $\langle E(L)^2\rangle_{HS} \sim (1 - g/g_2^{HS})^{-N}$ as $g \uparrow g_2^{HS}$ where $(g_2^{HS})^{-1} = 2\int_{-l_{HS}/2}^{l_{HS}/2} |C(x)|^2\, dx$ with $l_{HS} = \min(l_c, L)$. Beside the fact that g_2^{HS} is not equal to g_2 [5], the most striking discrepancy with the exact calculation is the degree of divergence. For large systems with $L \gg l_c$, the simplest HS model badly overestimates the exponent of the divergent factor. This incorrect asymptotic behavior is a direct consequence of the assumption of hot-spot statistical independence. Any nonvanishing hot-spot correlation will split the degeneracy of the N-fold pole $g = g_2^{HS}$, lowering the exponent of the divergent factor (down to the correct value 1 if the new smallest pole is simple). The necessity for any realistic HS model to include HS correlations is born out by the fact that maxima of a Gaussian field are more correlated than the field itself [9]. Thus, a satisfactory HS model should at least include both the HS field fluctuations (according to [5]) and the HS correlations, making it practically untractable. This striking contrast between the complexity of a proper HS approach and the simplicity of the exact calculation hints that the HS ansatz might not be adequate for large homogeneous systems.

Let us reconsider the problem without any *a priori* assumption on the statistical importance of hot-spots. From the solution to Eq. (1), it can be seen that the divergence of $\langle E(L)^2\rangle$ is due to the realizations of S with an arbitrarily large $||S||_2^2 \equiv Lu$ [10]. Do these particular realizations of S possess a strong coherent component leading to a well defined (nearly) deterministic profile? Does this profile look like a row of hot-spots? For any given $r > 0$, let S_r be the conditional field corresponding to the realizations of S with $Lu \geq r$. The realizations of S which determine the divergence of $\langle E(L)^2\rangle$ are also realizations of S_r, with arbitrarily large r. Thus, one has to determine whether S_r has a strong coherent component as $r \to +\infty$. Expanding S according to the Karhunen-Loève expansion [6], it can be shown that, for any given $r > 0$ and any realization of S, $d_2^2(r^{-1/2}S, \varphi_1 \mathrm{e}^{i\theta}) = \left(\sqrt{1 + \kappa_1\sigma/r} - 1\right)^2 + \Delta u/r$, where d_2 denotes the L^2-distance [10], φ_1 is the normalized eigenfunction associated with κ_1, $\theta = \arg(\varphi_1, S)$, $\sigma = s_1 - r/\kappa_1$, and $\Delta u = \sum_{n\geq 2} \kappa_n s_n$, where the s_ns are independent exponential random variables with $\langle s_n\rangle = 1$. From the probability distribution of s_n one finds that, assuming $Lu \geq r$, both σ and Δu remain bounded with probability one as $r \to +\infty$ [11]. Since the expression of $d_2^2(r^{-1/2}S, \varphi_1 \mathrm{e}^{i\theta})$ is true for any realization of S, it applies to any realization of S_r and one obtains

$$r^{-1/2} S_r(x) \to \varphi_1(x)\mathrm{e}^{i\theta} \qquad (r \to +\infty), \tag{2}$$

with probability one [12]. Accordingly, the realizations of S determining the divergence of $\langle E(L)^2\rangle$ do possess a well defined deterministic profile given by $\varphi_1(x)$. This profile results from the unbounded raise of the statistical weight of the φ_1-component of S as u tends to infinity. Since $\varphi_1(x)$ is a one-bump delocalized mode spreading over the whole interaction length, the emerging picture for $L \gg l_c$ is extremely different from the HS model ansatz of successive amplifications in localized structures of size $\sim l_c$ [13].

The asymptotic result (2) may be regarded as somewhat academical in the sense that, for any given realization of S, u will be finite with probability one. To check the validity of our scenario for finite u, we have reconsidered the problem (1) in which S is sampled from numerical realizations of a two-dimensional (2D) top-hat random-phase-plate field [14], S_{RPP}, with realistic parameters. We have simulated a 0.35 μm laser with a $f/8$ aperture propagating in a box of length 1 mm and cross section 2.24 mm. With these parameters the longitudinal and transverse correlation lengths of the smoothed laser field are $l_c = 100$ μm and $l_{c\perp} = 1.61$ μm. Consider the 2D problem where $E(y, x)$ is given by (1) with $S(x) = S_{RPP}(y, x)$. Let $S_{RPP,i}$ be the ith realization of S_{RPP}, $G_i(y) = g||S_{RPP,i}||_2^2(y)$ the gain factor at y and $x = L$, $\langle G\rangle = gL$, and $y_j = jl_{c\perp}$ with $1 \leq j \leq 1391$. We have used the following two samples: (i) $\{S\}_u \equiv \{S_{RPP,i}(y_j, x)\}$ such that $G_i(y_j) > u\langle G\rangle$ among 60000 realizations; and (ii) $\{S\}_u^{max} \equiv \{S_{RPP,i}(y_i^*, x)\}$ where y_i^* maximizes $G_i(y)$ and such that $G_i(y_i^*) > u\langle G\rangle$ among 60000 realizations. $\{S\}_u^{max}$ samples S_{RPP} behind the highest peak of $|E(y, L)|^2$, where S_{RPP} is expected to contribute effectively to the reflectivity near and above threshold. We have measured the emergence of φ_1 through: (a) the L^2-distance between S and its φ_1-component, $d_2(\hat{S}, \hat{S}_1)$, where $\hat{S} = S/||S||_2$ and $\hat{S}_1 = (\varphi_1, \hat{S})\varphi_1$; and (b) the relative contribution of the φ_1-component to the gain, $\alpha_1 \equiv G_1/G$, where $G = g||S||_2^2$ and $G_1 = g|(\varphi_1, S)|^2$. These two quantities are related to each other by $\alpha_1 = 1 - d_2^2(\hat{S}, \hat{S}_1)$.

Figure 1 shows the probability distribution of $d_2(\hat{S}, \hat{S}_1)$ estimated from the samples $\{S\}_0$ and $\{S\}_u^{max}$ with $u = 3$, 4, and 5. The last three curves are conditional probabilities knowing that $|E(y, L)|^2$ is maximum with $G/\langle G\rangle > 3$, 4, and 5 respectively. It can be seen that behind the peaks of $|E(y, L)|^2$, the x-profile of S_{RPP} is significantly closer to $\varphi_1(x)$ than average, and the larger the peak the smaller $d_2(\hat{S}, \hat{S}_1)$ as expected from Eq. (2). The statistical bias of $\hat{S}$ towards φ_1 could have been brought to the fore otherwise. For instance, (2) implies

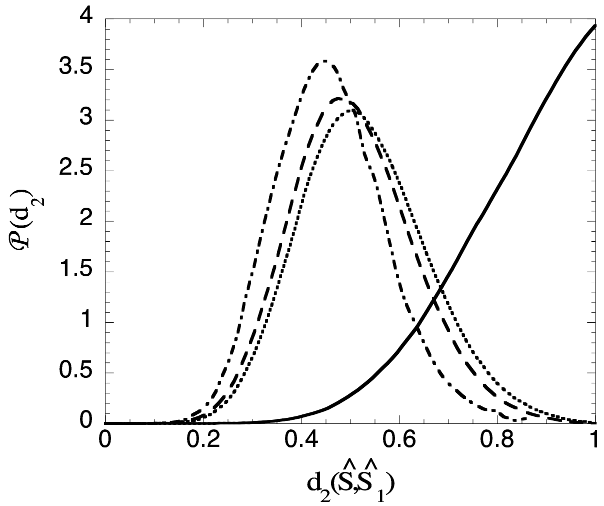

FIG. 1: Probability distribution of $d_2(\hat{S}, \hat{S}_1)$ estimated from $\{S\}_0$ (solid line), $\{S\}_3^{max}$ (dotted line), $\{S\}_4^{max}$ (dashed line), and $\{S\}_5^{max}$ (dashed-dotted line).

$\langle|\hat{S}(x)|^2\rangle_u^{1/2} \to |\varphi_1(x)|$ as $u \to +\infty$, where $\langle\cdot\rangle_u$ denotes conditional average knowing $G > u\langle G\rangle$. Figure 2 shows sample estimates of $\langle|\hat{S}(x)|^2\rangle^{1/2}$ for $\{S\}_0$, $\{S\}_4^{max}$, and $\{S\}_5^{max}$. The emergence of $|\varphi_1(x)|$ as u increases is obvious.

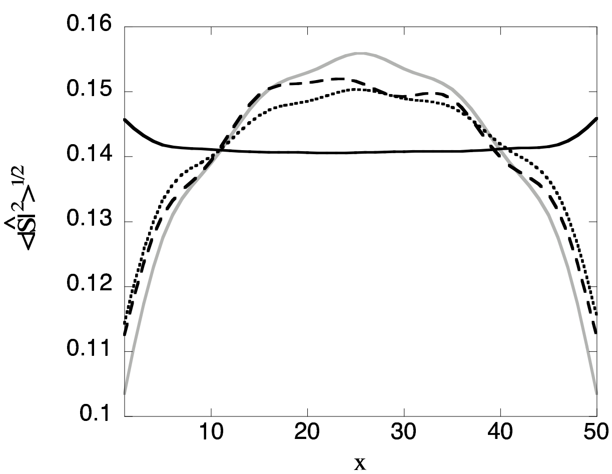

FIG. 2: Profiles of $|\varphi_1(x)|$ (grey line) and $\langle|\hat{S}(x)|^2\rangle^{1/2}$ estimated from $\{S\}_0$ (solid line), $\{S\}_4^{max}$ (dotted line), and $\{S\}_5^{max}$ (dashed line).

By themselves, the previous results are not sufficient to rule out the HS approach. If the relative HS contribution to the gain is found to be close to 1 for finite (realistic) u, there is no reason for not regarding the HS model as a satisfactory heuristic model capable of giving a good estimate of the reflectivity. To address this point, we have compared α_1 with its HS model counterpart

$$\alpha_{HS} \equiv \frac{G_{HS}}{G} = \frac{l_{HS}}{||S||_2^2}\left[1 - \frac{1}{3}\left(\frac{l_{HS}}{2l_c}\right)^2\right]\sum_{n=1}^{M}|S(x_n)|^2,$$

where G_{HS} is the HS contribution to G and x_n is the location of the nth local maxima of $|S(x)|^2$ with $|S(x_n)|^2 > 3$. It is important to notice that, since we make *no* assumption about HS statistical independence, effects of HS correlations [9] are automatically taken into account in our statistics of α_{HS}. Figures 3 and 4 show the probability distributions of α_1 and α_{HS} estimated from the samples $\{S\}_u - \{S\}_{u+0.12}$ as functions of (u, α_1) and (u, α_{HS}) respectively. These probabilities are conditional knowing $u < G/\langle G\rangle \leq u + 0.12$. For $u > 3$, $\{S\}_u^{max}$ yields similar results (there is no realization of $\{S\}_u^{max}$ with $G < 3\langle G\rangle$).

The behavior of α_1 consists of two phases: (A) a rapid increase up to $\alpha_1 \sim 0.8$ for $0 \leq u \lesssim 3$ corresponding to the slump of the number of not φ_1-dominated realizations as u increases; and (B) a slower growth for $u > 3$ where all the realizations are φ_1-dominated with (small) superimposed fluctuations the relative importance of which decreases slowly as u increases. The

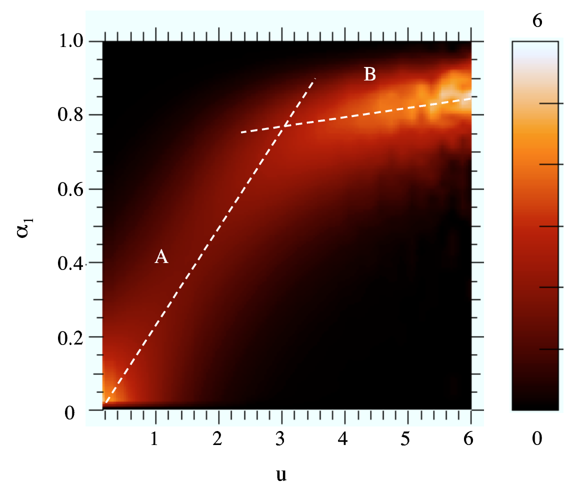

FIG. 3: Probability distribution of $\alpha_1 \equiv G_1/G$ knowing $u \equiv G/\langle G\rangle$ as a function of u and α_1. The dashed lines indicate the most probable value of α_1 at given u.

contribution of φ_1 represents more than 80% of the gain for u as small as $3 \lesssim u \lesssim 6$, which confirms the relevance of Eq. (2) for finite u and justifies its applicability even for low values of u.

The behavior of α_{HS} consists of four phases. The first three ones correspond to well defined populations of realizations: (A) realizations with no HS; (B) [resp. (C)] realizations with one (resp. two) HS. The saturation at $\alpha_{HS} \simeq 45\%$ for $u > 3$ (D), which is associated with rare realizations with 3 to 5 HS, does *not* depend on L for $L \gg l_c$. This remarkable result follows from the fact that, for large u and L/l_c, local maxima of $|S|^2$ correspond to

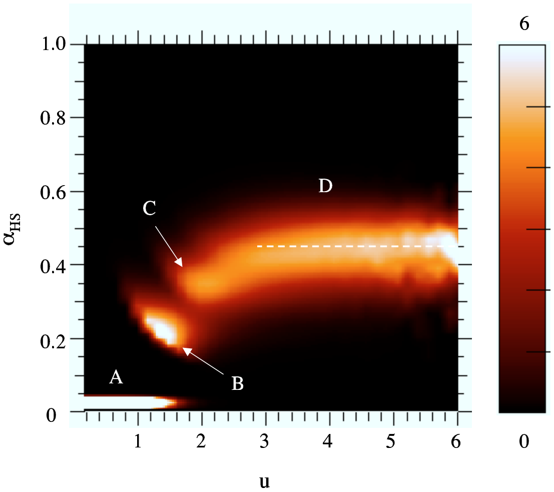

FIG. 4: Probability distribution of $\alpha_{HS} \equiv G_{HS}/G$ knowing $u \equiv G/\langle G\rangle$ as a function of u and α_{HS}. The dashed line at 45% indicates the most probable value of α_{HS} in the saturation phase (see the text for an explanation of this saturation).

small fluctuations of S around a large φ_1-component. In this limit, almost all of the $N \simeq L/2l_c$ possible maxima are HS (in the sense that $|S(x_n)|^2 > 3$), with width $\sim l_c$ and amplitude close to the one of the underlying φ_1-component. (Note that in this regime, hot-spots cannot be assumed to be statistically independent as there is a strong φ_1-induced HS correlation.) Thus, for large u and L/l_c, one has $\alpha_{HS} \sim N(l_{HS}/L)\alpha_1 \simeq 0.5\alpha_1 \lesssim 0.5$ independent of L. The latter inequality shows that, for large homogeneous systems with $L \gg l_c$ and as far as model (1) is concerned, HS contribution alone fails to provide a satisfactory estimate of the gain, in contradiction with the fundamental assumption of the HS approach.

In this Letter, the breakdown of the HS approach to convective amplification in large homogeneous systems has been established. By considering the simplest amplifier admiting a HS model description, it has been shown both analytically and numerically that high amplification does not originate in the hot-spots of the driver field, as assumed by the HS approach, but in a *delocalized* mode of this field spreading over the whole interaction length. We expect the above considerations to apply almost literally also to more realistic models of stationary convective amplification driven by a spatially smoothed laser beam [15]. The determination of the amplification taking diffraction into account is a particularly interesting problem. In this case, its seems that the only difference at lowest order consists in replacing $E(y, L)$, solution to (1) with $S(x) = S_{RPP}(y, x)$, by $\sum_n E_n(y, L)$ in which $E_n(y, L)$ is solution to (1) with $S(x) = S_{RPP}(y^{(n)}_{max}(x), x)$ where the $y^{(n)}_{max}$s are continuous paths maximizing the largest eigenvalue of $C(x, x') = \langle S_{RPP}(y(x), x)S_{RPP}(y(x'), x')^*\rangle$ with $y^{(n)}_{max}(L) = y$. Following then the same line of reasoning as above, one expects amplification to originate in thin tubes surrounding (or close to) the $y^{(n)}_{max}$s and along which the profile of S_{RPP} is given by the fundamental eigenmode of the corresponding $C(x, x')$. The verification of these predictions will be the subject of a future work.

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[11] The proof consists in showing that the conditional probability distributions $P(\Delta u|Lu \geq r)$ and $P(\sigma|Lu \geq r)$ do *not* depend on r as $r \to +\infty$.
[12] Going from L^2-convergence to pointwise convergence is allowed since all the stochastic fields in play are continuous with probability one.
[13] Note that in the somewhat pathological case where κ_1 is degenerate, $\varphi_1(x)e^{i\theta}$ will be replaced by the (normalized) projection of S_r upon the eigenspace of κ_1 which does not possess any well defined deterministic profile.
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